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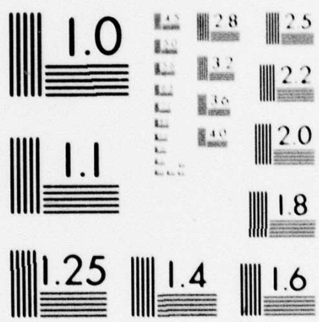

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Research Report CCS 329

A BI-EXTREMAL PRINCIPLE FOR ESTIMATING EFFICIENCY FRONTIER PARAMETER VALUES

by

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March 1979

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ABSTRACT

A new approach is supplied for evaluating the efficiency of decision making units, locating efficiency frontiers and estimating parameters from observational data. This is accomplished by means of a nonlinear-nonconvex bi-extremal principle which is subsequently shown to be essentially reducible to a finite sequence of linear programming problems. The development is illustrated by means of multiple output functions which are piecewise of Cobb-Douglas or general log linear type and which also allow for increasing, decreasing and constant returns to scale. The reduction of the bi-extremal principle to linear programming equivalence is also accomplished for much more general classes of functions.

1. INTRODUCTION

CCR [3] provided a nonlinear ratio extremal principle for determining a piecewise linear efficiency frontier for a collection of DMU's (Decision Making Units). This is fundamental to the subsequently developed procedures of Data Envelopment Analysis which are used to distinguish between "Program," "Managerial" and other types of efficiency. See [2]. One limitation of the preceding analysis, however, is that the efficiency frontiers are formed from functions which are of linear and/or piecewise linear type. Some situations may involve other, more general, parametric families of functions. One may then need to estimate the parameters with assurance that the resulting functions give the relevant efficiency frontiers.

Here we shall develop a bi-extremal principle for simultaneously achieving the parametric estimates and the associated efficiency measure to be assigned to each DMU. The basic idea is as follows: We "envelope" the observed values of the outputs by means of parametric functions of the observed input values. The envelope is "tightened" to rest on possible observed output values by means of a minimization operation. Then the parametric output functions are used to replace the observed output values in the manner of the nonlinear ratio maximization principle of CCR [3] for the efficiency determination of each DMU. Thus one obtains a bi-extremal principle of "maximim type."

In this paper, and for output functions of Cobb-Douglas type (or, more generally, for "ϕ linear" type)[1] we show that this nonlinear-nonconvex bi-extremal principle is, in fact, equivalent to a finite sequence of linear programming problems with one nonlinear vector operation. The linear programs further differ only in the right-hand side, thus greatly facilitating computation and analysis.

1] See section 6 below.

Most current approaches to the estimation of extreme values (e.g., as in mathematical statistics) are restricted to zero dimensional values (or frontiers) such as the end points of a uniform distribution. Our bi-extremal principle provides a new approach to the determination of multi-dimensional (extreme) frontiers such as are required for efficiency determination in the multiple input-multiple output situations that are commonly encountered in public policy evaluation problems. The principle is more general than such multiple input-multiple output applications, however, and extends to any case where extreme frontiers are of interest.

2. BACKGROUND

We follow the notation conventions of [3] and consider the commom input and output values for a collection of DMU's defined as follows:

$$(1)\qquad \begin{aligned} x_{ij} &= \text{the amount of input}, \quad i = 1, \ldots, m; \\ y_{rj} &= \text{the amount of output}, \quad r = 1, \ldots, s, \end{aligned}$$

where $j = 1, \ldots, n$ indexes each one of the DMU's being considered. These x_{ij} and y_{rj} values will generally represent observations generated from past behavior and we shall assume that they all have positive values.

The following formulation was given in [3] for determining the efficiency of any specified DMU_o from among this set of $j = 1, \ldots, n$ DMU's:

$$(2)\qquad \begin{aligned} &\max h_o = \frac{\sum_{r=1}^{s} w_r y_{ro}}{\sum_{i=1}^{m} u_i x_{io}} \\ &\text{subject to} \\ &\frac{\sum_{r=1}^{s} w_r y_{rj}}{\sum_{i=1}^{m} u_i x_{ij}} \le 1, \quad j = 1, \ldots, n \\ &w_r, u_i \ge 0, \ \forall\, i, r. \end{aligned}$$

Because the ratio in the functional also appears in the constraints we have max. $h_o = h_o^* \leq 1$ in any case and, as shown in [3], $h_o^* = 1$, iff DMU_o is efficient. Note, in particular, that this provides a scalar value for the wanted efficiency rating. [1]

This scalarization is achieved via the non-negative weights w_r, u_i assigned to the respective outputs and inputs. These weights are not assigned a priori in an arbitrary manner, however, but are determined objectively from the data as prescribed by (2). Hereafter we shall refer to them as "virtual weights " -- the intended analogy being to concepts like "virtual displacements" and/or "virtual work" in physics or engineering [2], which represent magnitudes that are not observed directly but are implicit, instead, in the underlying pnysical principles and models.

In the present paper we want to replace the formulation in (2) with another more general one that will, inter alia, enable us to estimate efficiency for Cobb-Douglas type (multiple output) functions. We want to do this, however, without losing contact with the developments in [3]. In particular, we want to preserve the Data Envelopment Analysis procedures of the preceding work so that we will thereby be able to effect evaluations that distinguish between programs or technologies and managerial efficiencies that are of importance for public sector guidance and control. See [2]. To help fix the ideas, we present our development in terms of Cobb-Douglas functions and then present the more general formulae.

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[1] A discussion of the operational significance of this rating as well as a transformation into linear programming equivalents (e.g. for computational efficiency) are provided in [ ].

[2] See, e.g., the discussion on p.647 in [1].

3. DEVELOPMENT

First, we introduce the "envelopment condition" on the outputs,

$$(3) \qquad y_{rj} \leq A_r \prod_{i=1}^{m} x_{ij}^{\mu_{ri}}, \qquad j = 1, \ldots, n,$$

where the variables μ_{ri} and A_r are constrained to be non-negative.[1] Here the symbol $\prod_{i=1}^{m}$ refers to the m-fold product of the $x_{ij}^{\mu_{ri}}$ values so that we are now restricting attention to functions which are of Cobb-Douglas form -- but extended to the case of multiple outputs $r = 1, \ldots, s$ in number.[2]

As already observed, we wish to make these inequalities as tight as possible. Hence we orient our objective toward a minimization over the A_r and μ_{ri} variables, as in the following,

$$\max_{w,u} \ \min_{A_r, \mu_{ri}} \ \frac{\sum_{r=1}^{s} w_r A_r \prod_{r=1}^{m} x_{io}^{\mu_{ri}}}{\sum_{i=1}^{m} u_i x_{io}}$$

(4) subject to

$$y_{rj} \leq A_r \prod_{i=1}^{m} x_{ij}^{\mu_{ri}}$$

$$\sum_{r=1}^{s} w_r y_{rj} \leq \sum_{i=1}^{m} u_i x_{ij}$$

$$A_r, \mu_{ri}, w_r, u_i \geq 0.$$

---

[1] Recall that y_{rj} and x_{ij} are observed positive constants.

[2] See [4] and [5] for further extension to the class of positive homogeneous and analytic functions for these Cobb-Douglas type of formulations.

This formulation provides our nonlinear-nonconvex bi-extremal principle, here formulated for functions of Cobb-Douglas (multiple output) type. Note that, unlike the development in [2], we here have a simultaneous determination of (i) the parameter values A_r, μ_{ri} and (ii) the efficiency ratings via (iii) the "virtual weights" w_r and u_i.

The formulation in (4) involves non-convexity (or non-concavity) in the bi-extremal objective. However, as noted in our Introduction, we shall bring it into an equivalent concave-convex form reducible to an *a priori* fixed finite sequence of linear programming problems with one nonlinear vector operation.

4. TRANSFORMATION

We now introduce the variables z_{ro} via

(5) $$A_r \prod_{i=1}^{m} x_{io}^{\mu_{ri}} \leq z_{ro} \, .$$

As a result of adjoining these additional constraints to (4) an equivalent objective function is

(6) $$\max_{w,u} . \ \min_{z_{ro}} . \ \frac{\sum_{r=1}^{s} w_r z_{ro}}{\sum_{i=1}^{m} u_i x_{io}} \, .$$

Next we make the following change of variables,

(7) $$\begin{aligned} w_r' &= t w_r \\ u_i' &= t u_i \\ t &\geq 0 \end{aligned}$$

such that

(8) $$\sum_{i=1}^{m} u_i' x_{io} = 1.$$

In these new variables we have a bilinear functional and constraints as follows

$$\max_{w',u'} \ \min_{z_{ro}} \ \sum_{r=1}^{s} w'_r z_{ro}$$

subject to

$$A_r \prod_{i=1}^{m} x_{io}^{\mu_{ri}} \leq z_{ro}$$

(9) $$y_{rj} \leq A_r \prod_{i=1}^{m} x_{ij}^{\mu_{ri}}$$

$$\sum_{r=1}^{s} w'_r y_{rj} \leq \sum_{i=1}^{m} u'_i x_{ij}$$

$$\sum_{i=1}^{m} u'_i x_{io} = 1$$

$$A_r,\ \mu_{ri},\ w',\ u' \geq 0.$$

By taking logs in the first two inequalities we can rewrite (9) in the form,

$$\max_{w',u'} \ \min_{z_{ro}} \ \sum_{r=1}^{m} w'_r z_{ro}$$

subject to

(10)
$$
\begin{aligned}
-\ln z_{ro} + \hat{A}_r + \sum_{i=1}^{m} \mu_{ri} \hat{x}_{io} & \leq 0 \\
- \hat{A}_r - \sum_{i=1}^{m} \mu_{ri} \hat{x}_{ij} & \leq -y_{rj} \\
\sum_{i=1}^{m} u'_i x_{ij} & = 1 \\
-\sum_{r=1}^{s} w'_r y_{rj} - \sum_{i=1}^{m} u'_i x_{ij} & \leq 0
\end{aligned}
$$

$$z_{ro},\ \hat{A}_r,\ \mu_{ri},\ w'_r,\ u'_i \geq 0$$

where the caret above a symbol denotes natural logarithm.

5. LINEAR PROGRAMMING EQUIVALENT

Notice in (10) that the w', u' and the z_{ro}, $\hat{A}_r$, μ_{ri} sets of variables occur in separate, independent sets of constraints. These two constraint systems restrict their separate variable sets to convex and feasible sets. Further, with the bilinear functional, the bi-extremal problem is equivalent to a saddle-value problem and such an equivalence would hold for much more general bi-extremal problems than (10). We shall pursue this generality in subsequent papers.

Here we note next that the z_{ro}, $\hat{A}_r$, μ_{ri} constraint system separates into independent systems with one system for each output index r. Thus for any set of the non-negative w_r' values, we wish to minimize each z_{ro} subject to its separate (r^{th}) constraint system. Setting $z_{ro} = \ln(z_{ro})$, the r^{th} problem would be:

$$\min.\ e^{\hat{z}_{ro}}$$

(10.1) subject to

$$\hat{z}_{ro} - \hat{A}_r - \sum_{i=1}^{m} \mu_{ri}\hat{x}_{io} \geq 0$$

$$\hat{A}_r + \sum_{i=1}^{m} \mu_{ri}\hat{x}_{ij} \geq \hat{y}_{rj}, \quad j=1, \ldots, n$$

$$\mu_{ri} \geq 0, \qquad i=1, \ldots, m.$$

Or, since minimizing $e^{\hat{z}_{ro}}$ is equivalent to minimizing $\hat{z}_{ro}$, we have the linear programming problem.

$$\min.\ \hat{z}_{ro}$$

(11) subject to

$$\hat{z}_{ro} - \hat{A}_r - \sum_{i=1}^{m} \mu_{ri}\hat{x}_{io} \geq 0$$

$$\hat{A}_r + \sum_{i=1}^{m} \mu_{ri}\hat{x}_{ij} \geq \hat{y}_{rj}, \quad j=1, \ldots, n$$

$$\mu_{ri} \geq 0, \quad i=1, \ldots, m.$$

Notice that the functional and matrix coefficient structure of this linear programming system is the same for all $r=1, \ldots, s$. Only the right hand side, the $\hat{y}_{rj}$, would change with r. If one were to solve the system via the dual linear programming problems, only the functional would change with r. One then knows a priori each functional to be employed. Thus an optimal basis for one value of r would be a feasible basis for any value of r.

On obtaining the optimal $\hat{z}_{ro} \equiv \hat{z}^*_{ro}$, we make the inverse (exponential) transformation to get z^*_{ro} and thereby secure the following linear programming problem to determine w'^*, u'^*, (hence w^*, u^* if desired):

$$\max. \sum_{r=1}^{s} w'_r z^*_{ro}$$

$$(12) \qquad \sum_{i=1}^{m} u'_i x_{io} = 1$$

subject to

$$\sum_{r=1}^{s} w'_r y_{rj} - \sum_{i=1}^{m} u'_i x_{ij} \leq 0$$

$$w'_r,\ u'_i \geq 0.$$

The maximal value of the functional $(\equiv \sum_r w^{*\prime} z^*_{ro})$ is then the efficiency rating for DMU_o. Similarly, the optimal values of the variables, $\hat{A}^*_r$, μ^*_{ri} give us the local Cobb-Douglas type envelope $A^*_r \prod_{i=1}^{m} x_{io}^{\mu^*_{ri}}$ function at this "0^{th}" DMU.

6 GENERALIZATION

It is clear that to get equivalent linear programming systems of the sort of (11) and (12) to our bi-extremal principles much more general output functions than those of Cobb-Douglas or general log linear type are permissible. We shall call an output function $g(\theta_1, \ldots, \theta_{m_1}, x_1, \ldots, x_{m_2})$ "ϕ-linear" iff:

$$(13) \qquad \phi[g(\theta, x)] = \sum_{k=1}^{m_1} \theta_k f_k(x)$$

where ϕ is a monotone strictly increasing function.

We shall assume each output function g_r is ϕ_r-linear. Applying the same techniques as in our Cobb-Douglas cases, we can reduce our bi-extremal principle to solution of the following two linear programming systems. First,

$$\min.\ \xi_{ro} \quad \sum_{k=1}^{m_1} \theta_{rk} f_{rk}(x^0)$$

(14.1)

$$\text{subject to} \quad \sum_{k=1}^{m_1} \theta_{rk} f_{rk}(x^j) \geq \phi_r(y^j) \quad , \ j = 1, \ldots, n,$$

plus any other relevant linear inequalities on the θ_{rk} which do not involve any other value of r. With min $\xi_{ro} = \xi^*_{ro}$ available, the efficiency system then becomes the second of these two linear programs -- viz.,

$$\max.\ \sum_{r=1}^{s} w'_r \phi_r^{-1}(\xi^*_{ro})$$

(14.2)

$$\text{subject to} \quad \sum_{i=1}^{m} u'_i x_{io} = 1$$

$$\sum_{r=1}^{s} w'_r y_{rj} - \sum_{i=1}^{m} u'_i x_{ij} \leq 0$$

$$w'_r,\ u'_i \geq 0 .$$

CONCLUSION

Between CCR [3] and the above, we have now explicitly covered the case of ϕ-linear functions -- which includes log linear and Cobb-Douglas families of functions. Convenience of use and the extent of their validation in empirical studies makes members of our Cobb-Douglas family of special interest. It should be noted, moreover, that our formulations extend to multiple outputs and to efficiency frontiers which are piecewise Cobb-Douglas. Also, we did not restrict the choice of μ_{ri} to the case of constant returns to scale. Hence, unlike the piecewise linear case of [3], we now permit increasing and/or decreasing returns to scale both in the frontiers and in the DMU's being evaluated.

The measures for efficiency, however, retain their operational significance. See [3]. It should be specifically noted that in the cases of increasing, decreasing and constant returns to scale the efficiency evaluation is effected by reference to the frontier segments that have this same property. In

other words, we can now ascertain whether a DMU that has increasing returns to scale is operating efficiently within any such zone as well as whether it is attaining scale economies. Moreover, we can also restrict particular coefficients so that they provide only one of increasing, decreasing or constant returns to scale. This could be done by introducting inequalities on the exponents or functions without altering any of the properties of the above models. Finally we should note that these multiple output/multiple input formulations permit these features to be used in <u>some</u> of the output/input relations without requiring them for <u>all</u> of the others at the same time.

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[2] __________, __________ and E. Rhodes, "An Application of the Method of Data Envelopment Analysis to Selected Program Follow Through Input and Output Variables," Research Report, Department of Environmental Science, State University of New York at Buffalo, NY Jan., 1979.

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